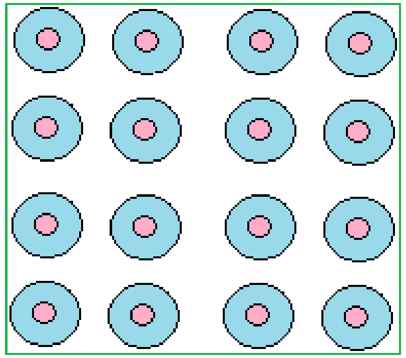
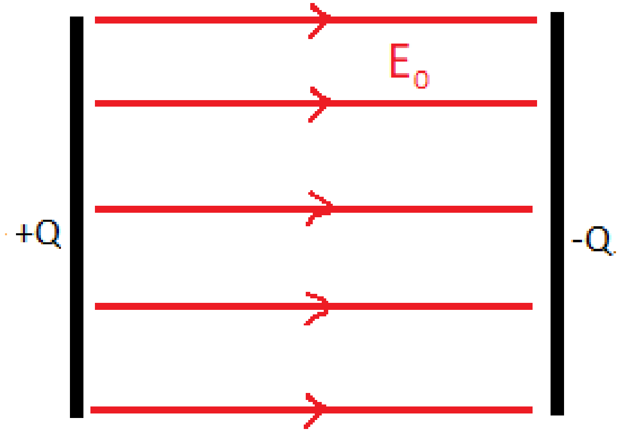
A.6 Electric Fields in Matter

One last thing before we jump into circuits. Having now covered force and energy, we'd (I'd?) now like to investigate what will happen when we place a typical chunk of matter in an electric field. So to that end,

Consider a chunk of matter, which consists, we'll suppose, of a lattice of atoms (pink guys nuclei, and blue clouds are electrons). And that the nucleus has charge q, and the electron cloud therefore –q. Further, we'll suppose that there are n atoms per unit volume, where n = (N_A/molar mass)·density.



And consider an electric field, E_0, perhaps created by charging two plates in equal and opposite fashion.



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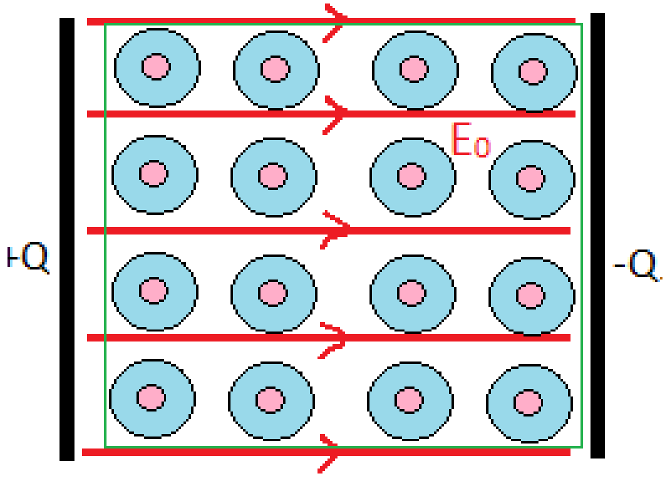
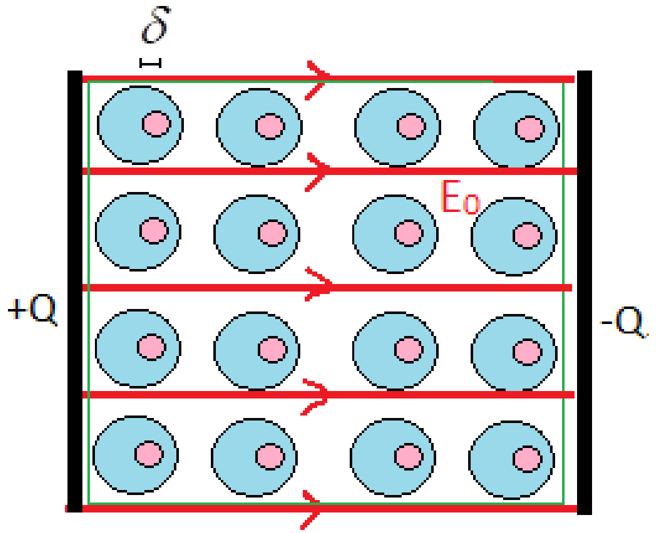
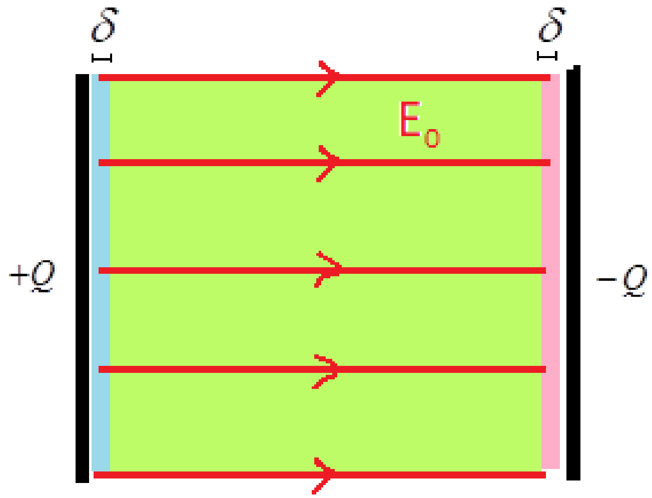
What happens when we put the matter in the field?



Well, the field will exert a force on the atoms: it will try to push the nuclei to the right, and pull the electron clouds to the left.



Which would result in a net negative accumulation of charge to the left, and net positive accumulation of charge to the right



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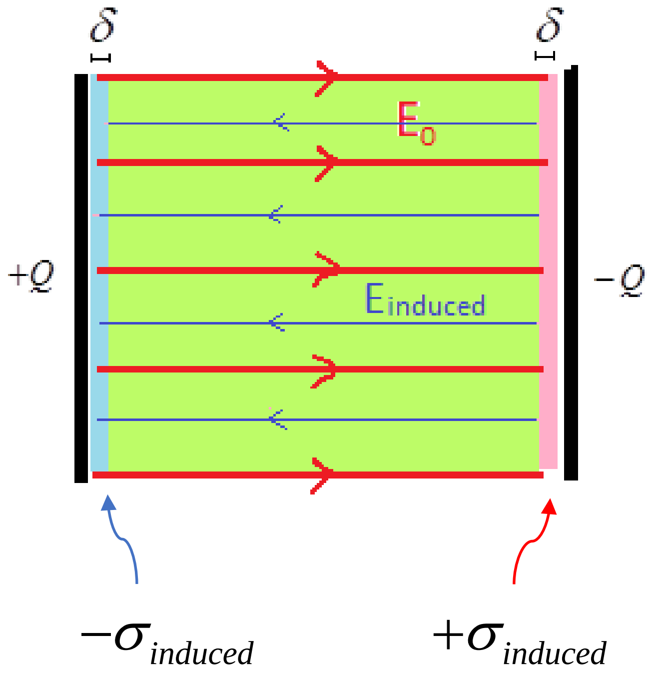
But now *another* electric field, $E_{induced}$, will thus be created, due to all the polarized atoms, and the consequent net charge sheets that accumulated on either end.



As we can see, it points opposite the original field, and so the net field will be reduced, at least within the material. When we study circuits, we'll want to know what this new field is. Might as well do it now,

$$E = E_0 - E_{induced}$$

$$= E_0 - 2\times\frac{|\sigma_{induced}|}{2\varepsilon_0}$$

$$= E_0 - 2\times\frac{\rho\delta}{2\varepsilon_0} \qquad \text{remember } \sigma = \rho\cdot\text{thickness of sheet}$$

$$= E_0 - \frac{nq\delta}{\varepsilon_0} \qquad \text{can write } \rho = nq$$

But to go any further requires knowing what δ is. So yeah let's do that....

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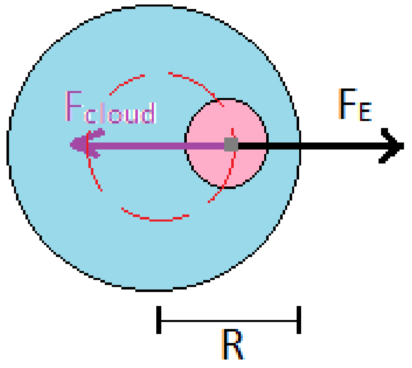
Remember δ is a thing because the ambient field is pushing the nucleus rightward, and the electron cloud leftward. It is, for instance, the distance the nucleus will shift in the electron cloud. It will shift to the point where the ambient field (**E**) force matches the internal electron cloud's field's force. So we need to know the strength of the electric field within the electron cloud. Let's say the electron cloud is uniform, out to some radius R.



$$E_{cloud} = \frac{kq_{enclosed}}{r^2} = \frac{k\left(q\dfrac{V_{enclosed}}{V_{cloud}}\right)}{\delta^2} = \frac{kq}{\delta^2}\left(\frac{\frac{4}{3}\pi\delta^3}{\frac{4}{3}\pi R^3}\right) = \frac{kq\delta}{R^3}$$

Now that we have E_{cloud}, we can determine where the forces on the nucleus will match.

$$\sum F = 0 \longrightarrow qE - q\left(\frac{kq\delta}{R^3}\right) = 0 \longrightarrow \delta = \frac{ER^3}{kq}$$

And now we can get the ambient field, E.

$$E = E_0 - \frac{nq\delta}{\varepsilon_0} \longrightarrow E = E_0 - \frac{nq}{\varepsilon_0}\frac{ER^3}{kq} \longrightarrow E = E_0 - 4\pi nR^3E \longrightarrow E = \frac{E_0}{1+4\pi nR^3}$$

But note: $4\pi nR^3 = 4\pi\frac{1}{V_{atom}}R^3 = 3\frac{V_{cloud}}{V_{atom}}$ so we can say: $E = \dfrac{E_0}{1+3V_{cloud}/V_{atom}}$

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So that's basically it. Just some definitions,….

$$E = \frac{E_0}{1 + 3V_{cloud} / V_{atom}}$$

Two definitions are made.

$$\boxed{\begin{aligned} \chi_e &= \text{electric susceptibility} = 3V_{cloud} / V_{atom} \\ \kappa_e &= \text{dielectric constant} = 1 + \chi_e \end{aligned}}$$

The susceptibility basically measures how easy it is, for the ambient field, E, to pull the nucleus and electron cloud apart, i.e., how *susceptible* they are to being pulled apart.

And so we can write the ambient field as:

$$E = \frac{E_0}{\kappa_e}$$

We can also go back and get the induced charge density

$$\sigma_{induced} = \rho\delta \;\; = nq \cdot \frac{ER^3}{kq} \;\; = 4\pi\varepsilon_0 nER^3 = \chi_e \varepsilon_0 E \longrightarrow \sigma_{induced} = \chi_e \varepsilon_0 E$$

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And one more thing. The energy stored inside our material takes two forms: one is electric potential energy due to the electric field **E** = **E**$_0$ + **E**$_{induced}$, residing within the material. The other is *elastic* potential energy that's stored within the atoms themselves by virtue of their stretching apart in the electric field (technically this is also a form of electric potential energy but never mind). When we discuss capacitors, as part of the circuits stuff, we'll want to know what this total potential energy is. We'll examine the potential energy *density*, first.

$$u_{total} = u_E + u_{atomic\ springs}$$

$$= \frac{\varepsilon_0}{2}E^2 + n\cdot PE_{atomic\,spring} \qquad n = \#\,\text{atoms/volume}$$

$$= \frac{\varepsilon_0}{2}E^2 + n\cdot\frac{1}{2}k_{spring}\delta^2$$

$$= \frac{\varepsilon_0}{2}E^2 + n\cdot\frac{1}{2}\frac{kq^2}{R^3}\delta^2 \qquad F_{cloud} = qE_{cloud} = q\left(\frac{kq\delta}{R^3}\right) = \overbrace{\frac{kq^2}{R^3}}^{k_{spring}}\cdot\delta$$

$$= \frac{\varepsilon_0}{2}E^2 + n\cdot\frac{1}{2}\frac{kq^2}{R^3}\left(\frac{ER^3}{kq}\right)^2 \qquad \delta = \frac{ER^3}{kq}$$

$$= \frac{\varepsilon_0}{2}E^2 + n\cdot\frac{R^3}{2k}E^2$$

$$= \frac{\varepsilon_0}{2}E^2 + \chi_e\frac{\varepsilon_0}{2}E^2 \qquad \chi_e = 4\pi nR^3$$

$$u_{total} = \frac{\varepsilon_0(1+\chi_e)}{2}E^2$$

$$u_{total} = \frac{\kappa_e\varepsilon_0}{2}E^2$$

$$PE_{total} = \int u_{total}\,dV$$

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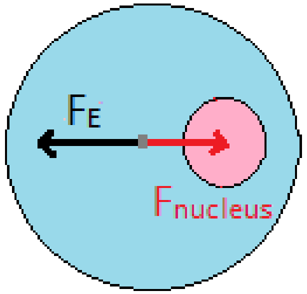
According to our (very rudimentary) formula, electric susceptibilities of non-polar molecules ought be something on the order of 3 or so, and the dielectric constant, differing only by 1, ought to be something around the same value. And this is what we find for a lot of materials. There are notable exceptions though. Following table I shamelessly copied from https://physics.info/dielectrics/, which seems to be a pretty cool physics learning site, if you want to check it out.

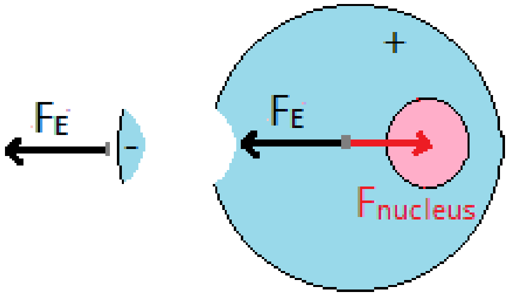
TABLE 26.1 Dielectric Constants and Dielectric Strengths of Various Materials at Room Temperature

Material	Dielectric Constant κ	Dielectric Strength[a] (V/m)
Air (dry)	1.000 59	3×10^6
Bakelite	4.9	24×10^6
Fused quartz	3.78	8×10^6
Neoprene rubber	6.7	12×10^6
Nylon	3.4	14×10^6
Paper	3.7	16×10^6
Polystyrene	2.56	24×10^6
Polyvinyl chloride	3.4	40×10^6
Porcelain	6	12×10^6
Pyrex glass	5.6	14×10^6
Silicone oil	2.5	15×10^6
Strontium titanate	233	8×10^6
Teflon	2.1	60×10^6
Vacuum	1.000 00	—
Water	80	—

[a] The dielectric strength equals the maximum electric field that can exist in a dielectric without electrical breakdown. Note that these values depend strongly on the presence of impurities and flaws in the materials.

The dielectric strength of a material is the field strengh which would be strong enough to pull an electron out of the cloud and off the atom entirely. This what happens during lightning.





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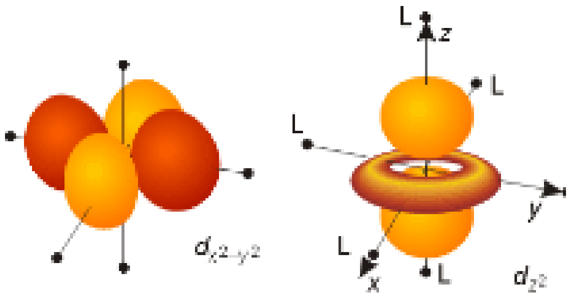
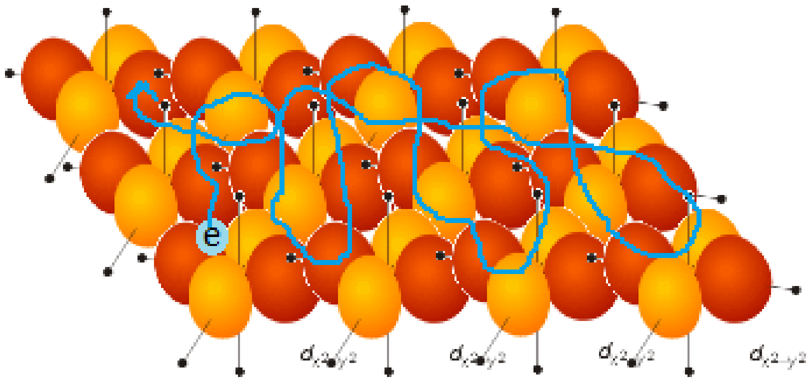
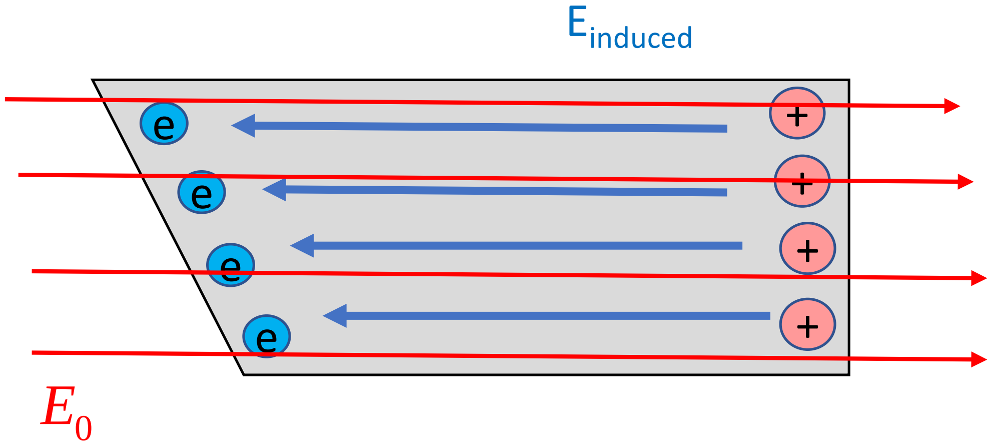
Now let's specialize our discussion to metals. Metals are exceptional because such an atom's valence electron(s) lie in the d or f orbitals which extend far beyond the nucleus. Because of this, neighboring atom's orbitals will overlap, and so the electrons will basically have space to roam the entire metal. And because of *this*, their susceptibility is basically infinite:





$$\chi_e = 3V_{cloud} / V_{atom}$$
$$= 3V_{metal} / V_{atom}$$
$$\sim 1\text{m}^3 / (1\text{nm})^3 \sim 10^{27} \sim \infty$$

1. And so when a metal is placed in an electric field, E_0, its electrons will be freely pulled to the opposite end of the metal, leaving the other end equally positive. This will keep happening until the field gets completely canceled inside the metal.



We know field will be entirely canceled since:

$$E = \frac{E_0}{\kappa_e} = \frac{E_0}{1+\chi_e}$$
$$= \frac{E_0}{1+\infty} = 0$$

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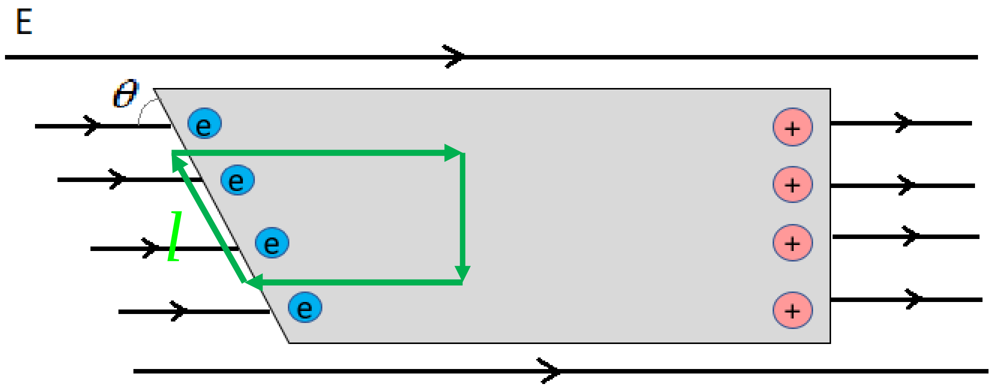
Gonna examine the field surrounding a metal just a bit more. We already know that it is 0 inside the metal.   What about outside?

2. The electric field near a metal's surface is perpendicular to the it, and therefore the metal's surface constitutes and equipotential.

We'll do a proof by contradiction. Suppose it's not perpendicular, but inclined at some angle. Then draw the following imaginary loop.



$$\Delta V_{loop} = El\cos\theta + 0 + 0 + 0$$

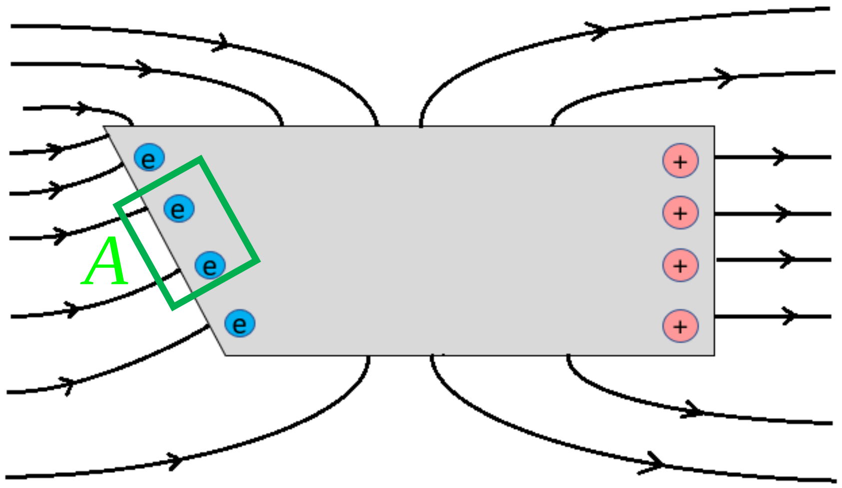
should always be zero for closed loop

$$0 = El\cos\theta$$

$$\theta = 90^\circ$$

3.   The strength of the electric field near a metal's surface is σ/ε_0

This time we'll use Gauss's law. We'll draw a Gaussian cube (use imagination) straddling the surface, which we'll suppose to have surface charge density σ.



$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enclosed}}{\varepsilon_0} \longrightarrow EA = \frac{\sigma A}{\varepsilon_0} \longrightarrow E = \frac{\sigma}{\varepsilon_0}$$

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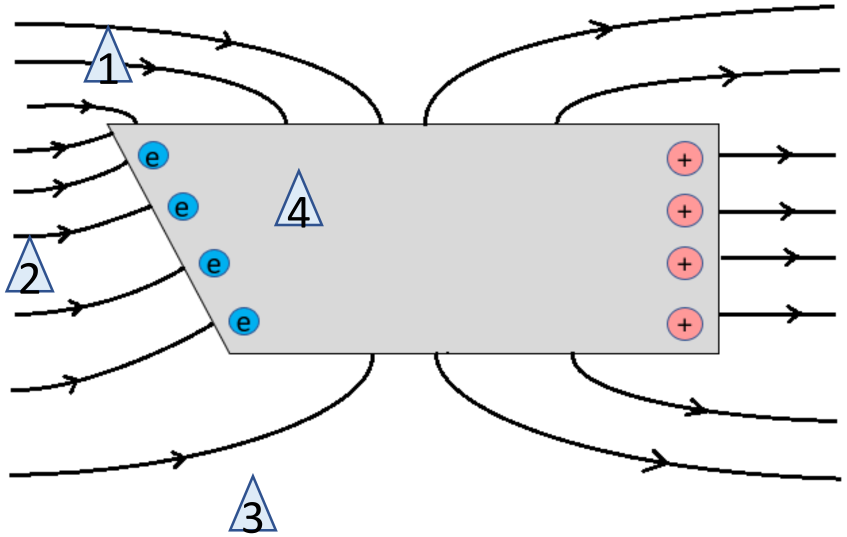
4. Somewhat counterintuitively, charges will congregate near the sharpest points of the metal, so σ, and E will be largest near the sharpest corners.



Consider the imaginary loop....

$$\Delta V_{loop} = \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4$$

$$\Delta V_1 = E_1 l_1 \qquad l_1 = \text{length of path 1}$$

$$\Delta V_2 = 0 \qquad \text{because it's perpendicular to field lines}$$

$$\Delta V_3 = -E_3 l_3 \qquad l_3 = \text{length of path 3}$$

$$\Delta V_4 = 0 \qquad \text{because } E = 0 \text{ inside metal}$$

again, should always be zero for closed loop

$$0 = E_1 l_1 - E_3 l_3$$

$$E_1 = \frac{l_3}{l_1} E_3 >> E_3$$

And since E = σ/ε_0, it follows that charges congregate near the edges too.

This is one reason lightning rods have sharp ends – to strengthen E, and encourage dielectric breakdown to occur *there*, where the resulting current can be safely grounded.

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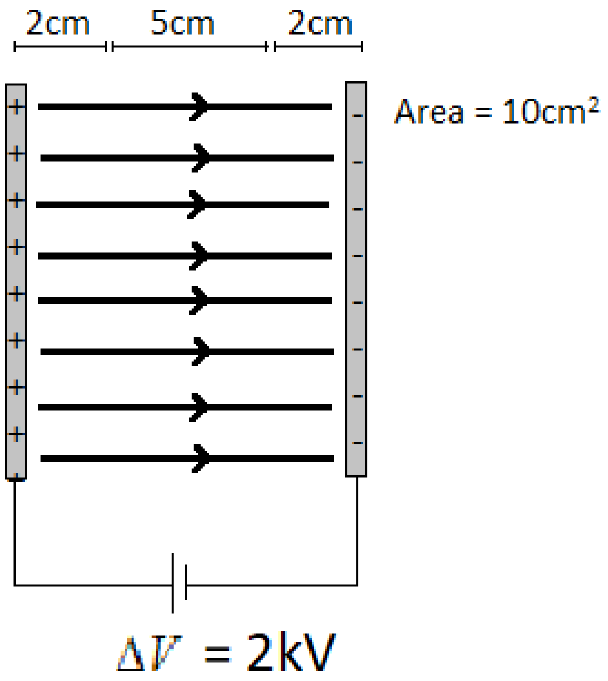


Say we've got a capacitor, hooked up to a battery 2kV battery. The capacitor consists of two parallel flat metallic plates with 10cm^2 area.

(a) What is the electric field between the plates?

$$\Delta V = -\int_{right\ plate}^{left\ plate} E dx = -E\int_{0.09}^{0} dx = E(0.09) \longrightarrow E = \frac{2000}{0.09} = 22\,\mathrm{kV/m}$$

(b) What is the energy stored between the plates?

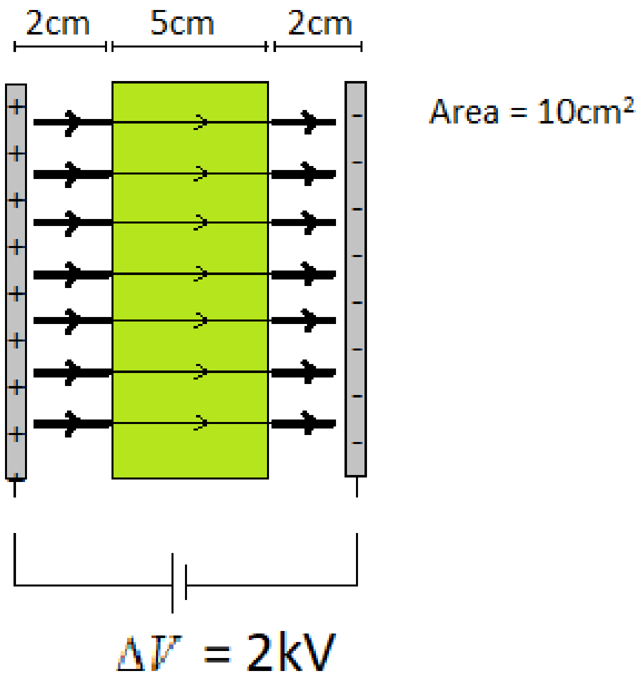
$$PE_E = \int u_E dV = \int \frac{\varepsilon_0}{2} E^2 dV = \frac{\varepsilon_0}{2} E^2 \int dV$$

$$= \frac{\varepsilon_0}{2} E^2 \cdot Volume$$

$$= \left[\frac{(8.85\times10^{-12})}{2}(22\times10^3)^2\right](0.10\cdot0.01^2)(0.09)$$

$$= 1.9\mathrm{nJ}$$

A.6 Electric Fields in Matter



(c) Say we disconnect the battery and place a strontium titanate slab in between the plates. What the field now?

Field is same in the air, but reduced in the dielectric by a factor of κ_e = 233 (look up in table). So field in the dielectric is:

$$E = \frac{E_0}{\kappa_e} = \frac{22\,\text{kV/m}}{233} = 94\,\text{V/m}$$

(d) What is the surface charge density on either side of the dielectric?

$$\sigma_{induced} = \chi_e \varepsilon_0 E = (\kappa_e - 1)\varepsilon_0 E = (233-1)(8.85\times10^{-12})(94) = 0.19\,\mu\text{C/m}^2$$

(e) What's the potential difference between the plates now?

$$\Delta V = -\int_{right}^{left} E dx$$

$$= -\int_{0.09}^{0.07} (22\text{kV/m})dx - \int_{0.07}^{0.02} (94\text{V/m})dx - \int_{0.02}^{0.0} (22\text{kV/m})dx$$

$$= -(22\text{kV/m})(-0.02\text{m}) - (94\text{V/m})(-0.05\text{m}) - (22\text{kV/m})(-0.02\text{m})$$

$$= 885\,\text{V}$$

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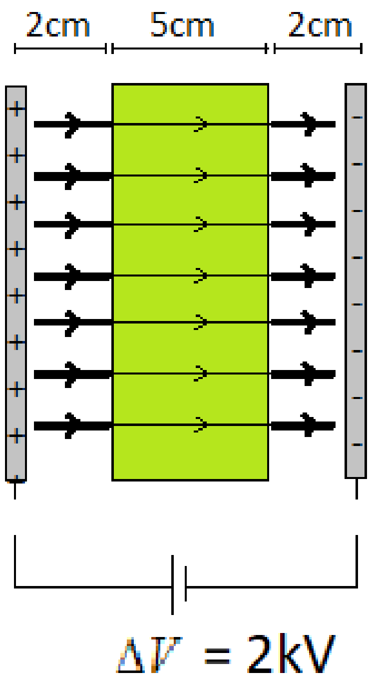

Area = 10cm²

(f) What's the new energy stored?

$$PE_{total} = \int u_{total} dV$$

$$= \frac{\varepsilon_0}{2} E_{air}^2 \cdot Volume_{air} + \frac{\kappa_e \varepsilon_0}{2} E_{strontium}^2 Volume_{strontium}$$

$$= 2\left[\frac{(8.85\times10^{-12})}{2}(22\times10^3)^2\right](0.10\cdot0.01^2)(0.02) + \left[\frac{(233)(8.85\times10^{-12})}{2}(94)^2\right](0.10\cdot0.01^2)(0.05)$$

$$= 0.86\,\text{nJ}$$

(g) The new PE is less than the original PE. So where does the missing energy go?

It actually goes into kinetic energy. The plates' field will pull the dielectric into itself; The kinetic energy the dielectric acquires during this process equals the difference in potential energy.

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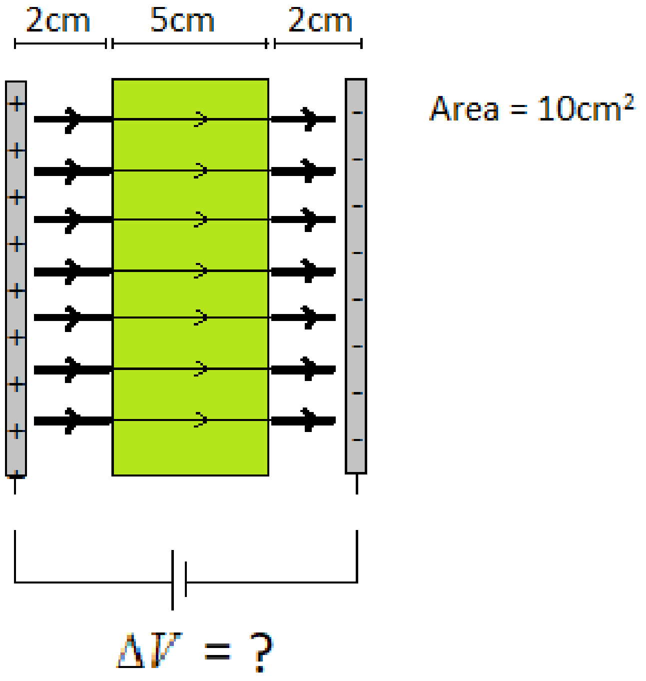

(h) Now say we hook the battery back up to the capacitor plates. What potential difference would induce dielectric breakdown in the strontium?

Evidently, according to the table, a field strength E = 8MV/m would start breakdown. So this is the field within the strontium. And then the field in the air would be: κE = (233)(8MV/m) = 1864 MV/m. So…

$$\Delta V = -\int_{right}^{left} E dx$$

$$= -\int_{0.09}^{0.07} (1864\,\text{MV/m})dx - \int_{0.07}^{0.02} (8\,\text{MV/m})dx - \int_{0.02}^{0.0} (1864\,\text{MV/m})dx$$

$$= -(1864\,\text{MV/m})(-0.02\text{m}) - (8\text{MV/m})(-0.05\text{m}) - (1864\text{MV/m})(-0.02\text{m})$$

$$= 75\,\text{MV}$$